

# ON THE $L^2$ -SOLUTIONS OF STOCHASTIC FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS; EXISTENCE, UNIQUENESS AND EQUIVALENCE OF SOLUTIONS

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**ABSTRACT.** The aim of this work is to prove existence and uniqueness of  $L^2$ -solutions of stochastic fractional partial differential equations in one spatial dimension. We prove also the equivalence between several notions of  $L^2$ -solutions. The Fourier transform is used to give meaning to SFPDEs. This method is valid also when the diffusion coefficient is random.

## 1. INTRODUCTION

Fractional calculus and stochastic analysis are connected concepts thanks to the selfsimilarity property. In recent years, mathematicians as well as physicians draw more attention to the use of the two topics simultaneously to model complex phenomena. Several definitions of fractional differential operators have been introduced based on probabilistic concepts, see for short list e.g. [15, 22, 23, 29]. Moreover, several phenomena, which are described to be anomalous, are modeled using fractional calculus and/or stochastic analysis, see e.g. [1, 2, 3, 4, 5, 6, 7, 12, 13, 16, 17, 29, 31, 32, 33, 34, 35] and the references therein. A phenomenon is described as anomalous if it is not covered by the Gaussian Markovian case. The anomaly is characterized by the long range dependence (LRD) effect and/or by the coexistence of the diffusive and the ballistic modes. One way to model the anomaly is to consider stochastic partial differential equations (shortly SPDEs) perturbed by non Gaussian noises, such as the Lévy or/and non Markovian noises, such as the fractional Brownian motion, see e.g. [21, 25, 26, 27] and others. The main difficulty in the study of SPDEs perturbed by non Markovian processes is due to the lack of a standard stochastic integral theory. To encounter this difficulty, SPDEs driven by fractional operator are used. Here the anomaly is presented via the Green function of the fractional operator, see e.g. [1, 5, 10, 11, 14, 31]. In these later works, authors are interested in the existence, uniqueness and the regularity of the solutions of different kinds of stochastic partial differential equations (SPEDs) driven by fractional operators. In [1], a linear SPED driven by the composition of the inverses of Riesz and Bessel potentials and perturbed by a space-time white noise is studied. In [10], the authors proved the existence and the uniqueness of the solution of an hyperbolic multidimensional SPDE driven by a power Laplacian and perturbed by a colored noise; white in time and homogeneous in space. The regularity of the solution is obtained in [11]. In [14], the authors considered high order stochastic fractional partial differential equations with entire derivatives and perturbed by

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space-time white noise. The non-Lipschitz case is treated in [5, 31]. In particular, the stochastic Burgers equation driven by fractional power of the Laplacian and perturbed by a cylindrical white noise respectively by a stable noise is studied. One of the main results, was the precision of the tree interaction between the dissipation, given by the fractional operator, the steepening, given by the nonlinear term and the regularity of the random noise.

The aim of this work is to prove existence and uniqueness of  $L^2$ -solutions of the SFPDEs introduced in [14]. The  $L^2$ -solution obtained in this paper coincides with the solution obtained in [14] under some special class of Lipschitz conditions. We present three different notions of  $L^2$ -solutions, mild, weak of first kind and weak of second kind. Moreover, we prove that these solutions are equivalent, for the literature on equivalence solutions, see e.g. [8, 24] for the evolutive SPDEs, [28] for the quasi-evolutive case and [18, 19] for the Walsh's approach. The result generalizes the equivalence obtained in [18, 20]. A special section is devoted to give meaning to SFPDEs using Fourier transform. This method is relevant when the diffusion coefficient is random and depends only on the spatial and the temporal variables but not on the solution. The study of the SFPDEs reduce to SDEs driven by martingales. The Fourier transform of the solution is a generalization of the Ornstein-Uhlenbeck process. We prove that the solution of the equation without derivatives of entire order given via the Fourier technique is equivalent to the mild solution.

The paper is organized as follows. In section 2, we prove existence and uniqueness of  $L^2$ -mild solutions. In section 3, we prove the equivalence of mild and weak solutions. In section 4, we apply the Fourier technique to define a solution for a special case of the equation studied. This notion of solution is equivalent to mild solution and to weak solutions.

We are interested in the following Cauchy problem:

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t}(t, x) = {}_x D_\delta^\alpha u(t, x) + \sum_{k=0}^m \frac{\partial^k h_k}{\partial x^k}(t, x, u(t, x)) + f(t, x, u(t, x)) \frac{\partial^2 W}{\partial t \partial x}(t, x), \\ t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u^0(x), \end{cases}$$

where  $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ ,  $m \in \mathbb{N}$ , such that  $1 \leq m \leq [\alpha]$ , where  $[\alpha]$  is the integer part of  $\alpha$  and  ${}_x D_\delta^\alpha$  is the fractional differential operator with respect to the spatial variable, to be defined below. We suppose that the functions  $f, g, h_k : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy Lipschitz and growth conditions:

for all  $T > 0$ , there exist a constant  $K_T > 0$  and functions  $a_k \in L^2(\mathbb{R})$ ,  $a_k \geq 0$ ,  $k = 0, 1, \dots, m+1$  such that for all  $t \in [0, T]$  and for all  $x, y, z \in \mathbb{R}$

$$(2) \quad \begin{aligned} & \left( |h_k(t, x, y) - h_k(t, x, z)| + |f(t, x, y) - f(t, x, z)| \right) \leq K_T |y - z|, \\ & |h_k(t, x, z)| \leq K_T (a_k(x) + |z|), \quad |f(t, x, z)| \leq K_T (a_{m+1}(x) + |z|). \end{aligned}$$

It is clear that when  $a_k \in L^2(\mathbb{R}) \cap L_\infty(\mathbb{R})$ , we find the Lipschitz conditions in [14]. Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and let  $W = \{W(t, x), t \geq 0, x \in \mathbb{R}\}$  be a centered Gaussian field defined on  $(\Omega, \mathcal{F}, P)$  with covariance function given by

$$K((t, x), (s, y)) = \frac{1}{4} (\text{sgn}(x) + \text{sgn}(y))^2 (t \wedge s) (|x| \wedge |y|),$$

where "sgn" denoted the sign function.  $W$  is in fact composed of two independent Brownian sheets, one in the positive direction of the spatial variable and the other one in the negative direction. Let  $(\mathcal{F}_t, t \geq 0)$  be an increasing and right-continuous filtration generated by  $W$ . The initial condition  $u^0$  is supposed to be a  $\mathcal{F}_0$ -measurable  $L^2(\mathbb{R})$ -valued function. We suppose that  $\alpha > 1$  and  $p \geq 1$ .

**Definition 1.** Let  $\alpha \in \mathbb{R}_+$ . The  $\alpha$ -fractional derivative operator is defined for all  $f \in \{g \in L^2(\mathbb{R}) / |\lambda|^\alpha \hat{g}(\lambda) \in L^2\}$ , by

$$(3) \quad D_\delta^\alpha f = \mathcal{F}^{-1}(\delta \psi_\alpha(\cdot) \hat{f}),$$

where  $|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$ ,  $[\alpha]_2$  is the largest even integer less than  $\alpha$  (even part of  $\alpha$ ) and  $\delta = 0$  when  $\alpha \in 2\mathbb{N} + 1$ ,

$$(4) \quad \delta \psi_\alpha(\lambda) = -|\lambda|^\alpha e^{-i\delta \frac{\pi}{2} \text{sgn} \lambda},$$

and  $\mathcal{F}^{-1}$  is the inverse Fourier transform on  $\mathbb{R}$  and  $\hat{f}$  is the Fourier transform of  $f$ .

The Fourier transform and its inverse are given by

$$(5) \quad \begin{aligned} \mathcal{F}\{f(x); \lambda\} &= \hat{f}(\lambda) = \int_{-\infty}^{+\infty} \exp(ix\lambda) f(x) dx, \\ \mathcal{F}^{-1}\{f(\lambda); x\} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ix\lambda) f(\lambda) d\lambda. \end{aligned}$$

The operator  $D_\delta^\alpha$  is the infinitesimal generator of an analytic semigroup of convolution given by the Green function  ${}_\delta G_\alpha(t, x) = \mathcal{F}^{-1}\{\exp[\delta \psi_\alpha(\lambda)t]; x\}$ . Hence it is closed densely defined operator. The function  ${}_\delta G_\alpha(t, x)$  is real but it is not symmetric relatively to  $x$ , when  $\delta \neq 0$ . Further, it is not everywhere positive when  $\alpha > 2$ . However,  $\int_{-\infty}^{+\infty} {}_\delta G_\alpha(t, x) dx = 1$ . The explicit form of  ${}_\delta G_\alpha(t, \cdot)$  is known only for  $\alpha \in \{\frac{1}{2}, 1, 2\}$ . Moreover,  ${}_\delta G_\alpha(t, \cdot)$  has a polynomial decrease when  $\alpha \notin \mathbb{N}$ . For more details on this operator and the properties of  ${}_\delta G_\alpha(t, \cdot)$  see [12, 13, 14]. In the following Lemma, we give some of the properties of the function  ${}_\delta G_\alpha(\cdot, \cdot)$  that we need in this context.

**Lemma 1.**

(i)  ${}_\delta G_\alpha(t, x)$  satisfies the semi-group property, or the Chapman Kolmogorov equation, i.e. for  $0 < s < t$

$${}_\delta G_\alpha(t + s, x) = \int_{-\infty}^{+\infty} {}_\delta G_\alpha(t, \xi) {}_\delta G_\alpha(s, x - \xi) d\xi,$$

(ii) For  $0 < \alpha \leq 2$ , the function  ${}_\delta G_\alpha(t, \cdot)$  is the density of a Lévy stable process in time  $t$ ,

(iii) For fixed  $t$ ,  ${}_\delta G_\alpha(t, \cdot) \in S^\infty = \{f \in C^\infty \text{ and } D_{\delta'}^\beta f \text{ is bounded and tends to zero when } |x| \text{ tends to } \infty, \forall \beta \in \mathbb{R}_+, |\delta'| \leq \min\{\beta - [\beta]_2, 2 + [\beta]_2 - \beta\} \text{ and } \delta' = 0 \text{ when } \beta \in 2\mathbb{N} + 1\}$ ,

(iv)  $\frac{\partial^l}{\partial x^l} {}_\delta G_\alpha(t, x) = t^{-\frac{l+1}{\alpha}} \frac{\partial^l {}_\delta G_\alpha}{\partial \xi^l}(1, \xi) \Big|_{\xi=t-\frac{1}{\alpha}x}$ , for all  $l \geq 0$  (when  $l=0$ , it is called the scaling property),

(v)  $\frac{\partial^l}{\partial x^l} {}_\delta G_\alpha(1, x) = \frac{1}{\pi} \sum_{j=1}^n |x|^{-\alpha j - (l+1)} \frac{(-1)^{j+l}}{j!} \Gamma(\alpha j + l + 1) \sin j \frac{(\alpha + \delta)}{2} \pi + O(|x|^{-\alpha(n+1) - (l+1)})$ , when  $|x|$  is large.

**Corollary 1.** Let  $\alpha > 1$ . For any fixed  $k \in \mathbb{N}$ , for  $\gamma > \frac{1}{\alpha + k + 1}$ ,

$$\left| {}_\delta G_\alpha^{(k)}(t, \cdot) \right|_\gamma = K_{\alpha, \gamma, k} t^{\frac{1 - (k+1)\gamma}{\alpha\gamma}}.$$

Let us also give the following stochastic Fubini's Theorem for Brownian sheet with respect to a deterministic non negative measure.

**Lemma 2.** (*Stochastic Fubini's Theorem*) Let  $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$  be a measure space and let  $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathcal{X} \rightarrow \mathbb{R}$  such that,  $\forall t > 0$ , the function  $f$  is  $\mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{X})$ -measurable and

$$\int_0^t \int_{\mathbb{R}} \mathbb{E} \left( \int_{\mathcal{X}} f(s, y, x) \mu(dx) \right)^2 dy ds < \infty.$$

Then the integrals

$$\int_0^t \int_{\mathbb{R}} \int_{\mathcal{X}} f(s, y, x) \mu(dx) W(dy ds), \quad \int_{\mathcal{X}} \int_0^t \int_{\mathbb{R}} f(s, y, x) W(dy ds) \mu(dx)$$

are well defined and are  $\mathbf{P}$ -a.s. equal.

Let  $L^p(\Omega, \mathcal{F}_0, L^2) = \{X : \Omega \rightarrow L^2(\mathbb{R}), X \text{ is } \mathcal{F}_0\text{-measurable and such that } \mathbb{E}|X|_2^p < \infty\}$ . The scalar product in  $L^2(\mathbb{R})$  is denoted by  $\langle \cdot, \cdot \rangle$  and the norm by  $|\cdot|_2$ . We note also that the value of the constants in this paper may change from line to line and some of the standing parameters are not always indicated. In particular, the dependence on  $T$ .

## 2. EXISTENCE AND UNIQUENESS OF SOLUTION

It is known that the equation (1) has no rigorous meaning. In the following definition, we give the notion of  $L^2$ -mild solution.

**Definition 2.** A  $L^2$ -valued  $\mathcal{F}_t$ -adapted stochastic process  $u = \{u(t, \cdot), t \in [0, T]\}$  is said to be a mild solution of the SFPDE in (1) on the interval  $[0, T]$ , with initial condition  $u^0$  if it satisfies the following integral equation for all  $t \in [0, T]$ ,

$$\begin{aligned} (6) \quad u(t, \cdot) = & \int_{\mathbb{R}} \delta G_{\alpha}(t, \cdot - y) u^0(y) dy \\ & + \sum_{k=0}^m (-1)^k \int_0^t \int_{\mathbb{R}} h_k(s, y, u(s, y)) \frac{\partial^k}{\partial z^k} \delta G_{\alpha}(t - s, z) |_{\cdot - y} dy ds \\ & + \int_0^t \int_{\mathbb{R}} f(s, y, u(s, y)) \delta G_{\alpha}(t - s, \cdot - y) W(dy ds). \end{aligned}$$

The equality in (6) is taken in  $L^2(\mathbb{R})$ .

**Theorem 1.** Let  $\alpha > 1$  and let  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$ , where  $p \geq 1$ . Then under conditions (2), the equation (1) admits a unique mild solution which satisfies the inequality

$$(7) \quad \sup_{[0, T]} \mathbb{E} |u(s)|_2^p < \infty.$$

The uniqueness is taken with respect to the norm in the left hand side of (7).

*Proof.*

Let  $\mathcal{H}$  be a Banach space of  $L^2$ -valued  $\mathcal{F}_t$ -adapted processes endowed by the norm

$$|u|_{\mathcal{H}}^p := \int_0^T e^{-\lambda t} \mathbb{E} |u(t)|_2^p dt < \infty,$$

where  $\lambda > 0$  will be determined later. Let  $\mathcal{H}^*$  denote the subspace of the processes of  $\mathcal{H}$  satisfying the assumption (7). We define on  $\mathcal{H}$  the operator  $\mathcal{A}$  by

$$(8) \quad \mathcal{A}u = \sum_{k=0}^{m+2} \mathcal{A}_k u,$$

where

$$\begin{aligned} \mathcal{A}_0 u(t) &= \int_{\mathbb{R}} \delta G_{\alpha}(t, \cdot - y) u^0(y) dy, \\ \mathcal{A}_{k+1} u(t) &= (-1)^k \int_0^t \int_{\mathbb{R}} h_k(s, y, u(s, y)) \frac{\partial^k \delta G_{\alpha}}{\partial z^k}(t - s, z)|_{\cdot - y} dy ds, \quad 0 \leq k \leq m \\ \mathcal{A}_{m+2} u(t) &= \int_0^t \int_{\mathbb{R}} f(s, y, u(s, y)) \delta G_{\alpha}(t - s, \cdot - y) W(dy ds). \end{aligned}$$

From the sequel it is easy to deduce that the operator  $\mathcal{A}$  takes  $\mathcal{H}$  to the space of  $\mathcal{F}_t$ -adapted processes  $\{u(t), t \geq 0\}$  such that for almost all  $t$ , we have  $u(t) \in L^2(\mathbb{R})$  a.s. We prove that  $\mathcal{H}^*$  is an invariant subspace for the operator  $\mathcal{A}$ . The restriction of  $\mathcal{A}$  on  $\mathcal{H}^*$  will be denoted by  $\mathcal{A}$  too. In fact, let  $u \in \mathcal{H}^*$ . It is easy to see that all the terms in the right hand side of (9), are  $\mathcal{F}_t$ -adapted processes when they exist. Further,

$\mathcal{A}_0 u(t) \in \mathcal{H}^*$  thanks to the assumption  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$  and to the inequality

$$(9) \quad |\mathcal{A}_0 u(t)|_2 = |G_{\alpha}(t, \cdot) * u^0|_2 \leq |G_{\alpha}(t, \cdot)|_1 |u^0|_2 \leq K |u^0|_2 \quad a.s.$$

For  $\mathcal{A}_k u(t)$ ,  $k = 0, 1, \dots, m$ , apply generalized Minkowsky's inequality, Young's inequality, corollary 1 and conditions (2) to get

$$\begin{aligned} \mathbb{E} |\mathcal{A}_{k+1} u(t)|_2^p &= \mathbb{E} \left( \left| \int_0^t \int_{\mathbb{R}} h_k(s, u(s)) * \delta G_{\alpha}^{(k)}(t - s, x - \cdot) ds \right|^2 dx \right)^{\frac{p}{2}} \\ &\leq \mathbb{E} \left( \int_0^t \left\{ \int_{\mathbb{R}} |h_k(s, u(s)) * \delta G_{\alpha}^{(k)}(t - s, x - \cdot)|^2 dx \right\}^{\frac{1}{2}} ds \right)^p \\ &\leq \mathbb{E} \left( \int_0^t |h_k(s, u(s))|_2 |\delta G_{\alpha}^{(k)}(t - s, \cdot)|_1 ds \right)^p \\ &\leq K \mathbb{E} \left( \int_0^t (t - s)^{-\frac{k}{\alpha}} (|a_k|_2 + |u(s)|_2) ds \right)^p \\ &\leq K \mathbb{E} \left( 1 + \int_0^t (t - s)^{-\frac{k}{\alpha}} |u(s)|_2^p ds \right) \\ &\leq K \left( 1 + \int_0^t (t - s)^{-\frac{k}{\alpha}} \mathbb{E} |u(s)|_2^p ds \right). \end{aligned}$$

(10)

This proves, on one hand that  $\sup_{[0, T]} \mathbb{E} |\mathcal{A}_{k+1} u(t)|_2^p < \infty$  i.e.  $\{\mathcal{A}_{k+1} u(t), t \geq 0\}$  is an  $L^2$ -valued process and satisfies (7). On the other hand, thanks to Fubini's Theorem, we get

$$\begin{aligned} \int_0^T e^{-\lambda t} \mathbb{E} |\mathcal{A}_{k+1} u(t)|_2^p dt &\leq K \int_0^T e^{-\lambda t} \left( 1 + \int_0^t (t - s)^{-\frac{k}{\alpha}} \mathbb{E} |u(s)|_2^p ds \right) dt \\ &\leq K \left( 1 + \int_0^T \left( \int_0^{T-s} e^{-\lambda \tau} \tau^{-\frac{k}{\alpha}} d\tau \right) e^{-\lambda s} \mathbb{E} |u(s)|_2^p ds \right) \\ (11) \quad &\leq K \left( 1 + \int_0^T e^{-\lambda s} \mathbb{E} |u(s)|_2^p ds \right) < \infty. \end{aligned}$$

Therefore,  $\mathcal{A}_{k+1}u \in \mathcal{H}^*$ ,  $\forall k = 0, 1, \dots, m$ . To estimate the stochastic integral, we use the factorization method (see [9]). By the semigroup property and the Fubini's Theorem and the identity

$$\int_{\sigma}^t (t-s)^{\beta-1} (s-\sigma)^{-\beta} ds = \frac{\pi}{\sin \pi \beta}, \quad \sigma \leq s \leq t, \quad 0 < \beta < 1,$$

we get the following representation of  $\mathcal{A}_{m+2}u(t, x)$

$$\mathcal{A}_{m+2}u(t, x) = \frac{\sin \pi \beta}{\pi} \int_0^t (t-s)^{\beta-1} \int_{\mathbb{R}} \delta G_{\alpha}(t-s, x-y) Y(s, y) dy ds,$$

where

$$Y(s, y) = \int_0^s (s-\sigma)^{-\beta} \int_{\mathbb{R}} \delta G_{\alpha}(s-\sigma, y-z) f(\sigma, z, u(\sigma, z)) W(dz d\sigma).$$

Let  $\frac{1}{\alpha p} - \frac{1}{2\alpha} < \beta < 1$ . By the inequalities cited above and the corollary 1, we get

$$\begin{aligned} \mathbb{E}|\mathcal{A}_{m+2}u(t)|_2^p &= \mathbb{E}\left(\int_{\mathbb{R}} \left|\frac{\sin \pi \beta}{\pi} \int_0^t (t-s)^{\beta-1} Y(s, \cdot) * \delta G_{\alpha}(t-s, \cdot) ds\right|^2 dx\right)^{\frac{p}{2}} \\ &\leq \mathbb{E}\left(\frac{|\sin \pi \beta|}{\pi} \int_0^t (t-s)^{\beta-1} |Y(s, \cdot) * \delta G_{\alpha}(t-s, \cdot)|_2 ds\right)^p \\ &\leq \mathbb{E}\left(\frac{|\sin \pi \beta|}{\pi} \int_0^t (t-s)^{\beta-1} |Y(s)|_p |\delta G_{\alpha}(t-s, \cdot)|_{\frac{2p}{3p-2}} ds\right)^p \\ &\leq K \mathbb{E}\left(\int_0^t (t-s)^{\beta-1+\frac{1}{2\alpha}-\frac{1}{\alpha p}} |Y(s)|_p ds\right)^p \\ &\leq K \int_0^t (t-s)^{\beta-1+\frac{1}{2\alpha}-\frac{1}{\alpha p}} \mathbb{E}|Y(s)|_p^p ds. \end{aligned}$$

(12)

On the other hand, under the condition  $0 < \beta < \frac{1}{2} + \frac{1}{\alpha p} - \frac{1}{\alpha}$  and using Burkholder-Davis-Gundy inequality and the generalized Minkowsky's inequality, Young's inequality, corollary 1, conditions (2) and Hölder inequality, we get for all  $0 \leq s \leq T$

$$\begin{aligned} \mathbb{E}[|Y(s)|_p^p] &\leq \int_{\mathbb{R}} \mathbb{E} \sup_{0 \leq \tau \leq s} \left| \int_0^{\tau} \int_{\mathbb{R}} (s-\sigma)^{-\beta} \delta G_{\alpha}(s-\sigma, y-z) f(\sigma, z, u(\sigma, z)) W(dz d\sigma) \right|^p dy \\ &\leq K \int_{\mathbb{R}} \mathbb{E} \left| \int_0^s \int_{\mathbb{R}} (s-\sigma)^{-2\beta} \delta G_{\alpha}^2(s-\sigma, y-z) f^2(\sigma, z, u(\sigma, z)) dz d\sigma \right|^{\frac{p}{2}} dy \\ &\leq K \mathbb{E} \int_{\mathbb{R}} \left| \int_0^s (s-\sigma)^{-2\beta} (\delta G_{\alpha}^2(s-\sigma, \cdot) * f^2(\sigma, \cdot, u(\sigma, \cdot)))(y) d\sigma \right|^{\frac{p}{2}} dy \\ &\leq K \mathbb{E} \left( \int_0^s (s-\sigma)^{-2\beta} |\delta G_{\alpha}^2(s-\sigma, \cdot) * f^2(\sigma, \cdot, u(\sigma, \cdot))|_{\frac{p}{2}} d\sigma \right)^{\frac{p}{2}} \\ &\leq K \mathbb{E} \left( \int_0^s (s-\sigma)^{-2\beta} |\delta G_{\alpha}^2(s-\sigma, \cdot)|_{\frac{p}{2}} |f^2(\sigma, \cdot, u(\sigma, \cdot))|_1 d\sigma \right)^{\frac{p}{2}} \\ &\leq K \mathbb{E} \left( \int_0^s (s-\sigma)^{-2\beta+\frac{2}{\alpha p}-\frac{2}{\alpha}} (|a_{m+1}|_2^2 + |u(\sigma)|_2^2) d\sigma \right)^{\frac{p}{2}} \\ &\leq K \left( 1 + \int_0^s (s-\sigma)^{-2\beta+\frac{2}{\alpha p}-\frac{2}{\alpha}} \mathbb{E}|u(\sigma)|_2^p d\sigma \right). \end{aligned}$$

Replacing this last inequality in (12), we get

$$\begin{aligned}
\mathbb{E}|\mathcal{A}_{m+2}u(t)|_2^p &\leq K \int_0^t (t-s)^{\beta-1+\frac{1}{2\alpha}-\frac{1}{\alpha p}} \mathbb{E}|Y(s)|_p^p ds \\
&\leq K \left(1 + \int_0^t (t-s)^{\beta-1+\frac{1}{2\alpha}-\frac{1}{\alpha p}} \left( \int_0^s (s-\sigma)^{-2\beta+\frac{2}{\alpha p}-\frac{2}{\alpha}} \mathbb{E}|u(\sigma)|_2^p d\sigma \right) ds \right) \\
&\leq K \left(1 + \int_0^t (t-\sigma)^{-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}} \mathbb{E}|u(\sigma)|_2^p d\sigma \right).
\end{aligned}
\tag{13}$$

But  $-\beta - \frac{3}{2\alpha} + \frac{1}{\alpha p} > -1$  because  $1 - \frac{3}{2\alpha} + \frac{1}{\alpha p} > \frac{1}{2} + \frac{1}{\alpha p} - \frac{1}{\alpha}$ . Hence  $\{\mathcal{A}_{m+2}u(t), t \geq 0\}$  is an  $L^2$ -process satisfying (7). Further,

$$\int_0^T e^{-\lambda t} \mathbb{E}|\mathcal{A}_{m+2}u(t)|_2^p dt \leq K(1 + \int_0^T e^{-\lambda \sigma} \mathbb{E}|u(\sigma)|_2^p d\sigma) < \infty.$$

It is clear that  $\frac{1}{\alpha p} - \frac{1}{2\alpha} < \frac{1}{2} + \frac{1}{\alpha p} - \frac{1}{\alpha}$ , provided that  $\alpha > 1$ . The parameter  $\beta$  satisfies the inequality  $\max\{0, \frac{1}{\alpha p} - \frac{1}{2\alpha}\} < \beta < \min\{\frac{1}{2} + \frac{1}{\alpha p} - \frac{1}{\alpha}, 1\}$ . This achieves the proof that  $\mathcal{A}u \in \mathcal{H}^*$ .

To prove the existence and the uniqueness of the solution in  $\mathcal{H}^*$ , we use the fixed point method. Let  $u, v \in \mathcal{H}^*$ , we have for all  $k = 0, 1, \dots, m$ , for all  $t > 0$ ,

$$\begin{aligned}
\mathbb{E}|\mathcal{A}_{k+1}u(t) - \mathcal{A}_{k+1}v(t)|_2^p &= \mathbb{E} \left( \left| \int_{\mathbb{R}} \int_0^t \delta G_{\alpha}^{(k)}(t-s, \cdot) * (h_k(s, u(s)) - h_k(s, v(s))) ds \right|^2 dx \right)^{\frac{p}{2}} \\
&\leq \mathbb{E} \left( \int_0^t |\delta G_{\alpha}^{(k)}(t-s, \cdot) * (h_k(s, u(s)) - h_k(s, v(s)))|_2 ds \right)^p \\
&\leq \mathbb{E} \left( \int_0^t |\delta G_{\alpha}^{(k)}(t-s, \cdot)|_1 |h_k(s, u(s)) - h_k(s, v(s))|_2 ds \right)^p \\
&\leq K \mathbb{E} \left( \int_0^t (t-s)^{-\frac{k}{\alpha}} |u(s) - v(s)|_2 ds \right)^p \\
&\leq K \int_0^t (t-s)^{-\frac{m}{\alpha}} \mathbb{E}|u(s) - v(s)|_2^p ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mathcal{A}_{k+1}u - \mathcal{A}_{k+1}v|_{\mathcal{H}}^p &\leq K \int_0^T e^{-\lambda s} \left[ \int_0^T e^{-\lambda \tau} \tau^{-\frac{m}{\alpha}} d\tau \right] \mathbb{E}|u(s) - v(s)|_2^p ds \\
&\leq K \lambda^{\frac{m-\alpha}{\alpha}} \Gamma\left(\frac{\alpha-m}{\alpha}\right) |u - v|_{\mathcal{H}}^p.
\end{aligned}
\tag{14}$$

For the stochastic integral, we use again the factorization method for the same  $\beta$ . We get

$$\mathcal{A}_{m+2}u(t, x) - \mathcal{A}_{m+2}v(t, x) = \frac{\sin \pi \beta}{\pi} \int_0^t (t-s)^{\beta-1} \int_{\mathbb{R}} \delta G_{\alpha}(t-s, x-y) \zeta(s, y) dy ds,$$

where

$$\zeta(s, y) = \int_0^s (s-\sigma)^{-\beta} \int_{\mathbb{R}} \delta G_{\alpha}(s-\sigma, y-z) (f(\sigma, z, u(\sigma, z)) - f(\sigma, z, v(\sigma, z))) W(dz d\sigma).$$

Using the calculus above, we obtain

$$\mathbb{E}|\mathcal{A}_{m+2}u(t) - \mathcal{A}_{m+2}v(t)|_2^p \leq K \int_0^t (t-s)^{\beta-1-\frac{1}{\alpha}+\frac{3p-2}{2\alpha p}} \mathbb{E}|\zeta(s)|_p^p ds,$$

and

$$\begin{aligned} \mathbb{E}|\zeta(s)|_p^p &\leq K \mathbb{E} \left( \int_0^s (s-\sigma)^{-2\beta} \left| \delta G_\alpha^2(s-\sigma, \cdot) \right|_{\frac{p}{2}} \left| (f(\sigma, \cdot, u(\sigma, \cdot)) - f(\sigma, \cdot, v(\sigma, \cdot)))^2 \right|_1 d\sigma \right)^{\frac{p}{2}} \\ &\leq K \int_0^s (s-\sigma)^{-2\beta+\frac{2}{\alpha p}-\frac{2}{\alpha}} \mathbb{E}|u(\sigma) - v(\sigma)|_2^p d\sigma. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}|\mathcal{A}_{m+2}u(t) - \mathcal{A}_{m+2}v(t)|_2^p &\leq K \int_0^t (t-s)^{\beta-1-\frac{1}{\alpha}+\frac{3p-2}{2\alpha p}} \int_0^s (s-\sigma)^{-2\beta+\frac{2}{\alpha p}-\frac{2}{\alpha}} \mathbb{E}|u(\sigma) - v(\sigma)|_2^p d\sigma ds \\ &\leq K \int_0^t (t-\sigma)^{-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}} \mathbb{E}|u(\sigma) - v(\sigma)|_2^p d\sigma, \end{aligned}$$

and therefore

$$\begin{aligned} |\mathcal{A}_{m+2}u - \mathcal{A}_{m+2}v|_{\mathcal{H}}^p &\leq K \int_0^T e^{-\lambda s} \left[ \int_0^T e^{-\lambda \tau} \tau^{-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}} d\tau \right] \mathbb{E}|u(s) - v(s)|_2^p ds \\ &\leq K \lambda^{\beta-\frac{1}{\alpha p}+\frac{3}{2\alpha}-1} \Gamma(1-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}) |u-v|_{\mathcal{H}}^p. \end{aligned}$$

(15)

For a good choice of the constant  $\lambda$  such that  $\max\{(m+1)K\lambda^{\beta-\frac{1}{\alpha p}+\frac{3}{2\alpha}-1}\Gamma(1-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}), K\lambda^{\frac{m-\alpha}{\alpha}}\Gamma(\frac{\alpha-m}{\alpha})\} < 1$ , the operator  $\mathcal{A}$  is then a contraction. This choice is possible because the exponent  $\beta - \frac{1}{\alpha p} + \frac{3}{2\alpha} - 1$  is also negative. So there exists an unique  $L^2(\mathbb{R})$ -valued  $\mathcal{F}_t$ -adapted process  $u \in \overline{\mathcal{H}^*} \subset \mathcal{H}$  solution of Equation (6), where  $\overline{\mathcal{H}^*}$  is the closure of the subspace  $\mathcal{H}^*$  in  $\mathcal{H}$ . We prove now that  $u \in \mathcal{H}^*$  i.e.  $u$  satisfies (7). In fact, we have on one hand  $\mathcal{A}u = u$ , on the other hand by a similar calculus of that done above, we get for all  $t > 0$ , the inequalities (9), (10) and (13). Hence

$$\begin{aligned} \mathbb{E}|u(t)|_2^p &\leq K \left( 1 + \int_0^t \left[ \sum_{k=0}^m (t-\sigma)^{-\frac{k}{\alpha}} + (t-\sigma)^{-\beta+\frac{1}{\alpha p}-\frac{3}{2\alpha}} \right] \mathbb{E}|u(\sigma)|_2^p d\sigma \right) \\ &\leq K \left( 1 + \int_0^t (t-\sigma)^{-\gamma} \mathbb{E}|u(\sigma)|_2^p d\sigma \right), \end{aligned}$$

(16)

where  $\gamma = \min\{\frac{m}{\alpha}, \beta - \frac{1}{\alpha p} + \frac{3}{2\alpha}\}$ . By Gronwall Lemma we get  $\mathbb{E}|u(t)|_2^p \leq K_\gamma e^{K_\gamma T}$ , hence  $u$  satisfies (7). We prove now the uniqueness of the solution with respect to the sup norm. Let  $u_1, u_2$  be two solutions of (6) in  $\mathcal{H}^*$ . By a similar calculus as above, we obtain for all  $t > 0$

$$\mathbb{E}|u_1(t) - u_2(t)|_2^p \leq K_\alpha \int_0^t \mathbb{E}|u_1(\sigma) - u_2(\sigma)|_2^p d\sigma.$$

Again by Gronwall Lemma we get  $\sup_{[0,T]} \mathbb{E}|u_1(t) - u_2(t)|_2^p = 0$  ( see e.g. [30] page 314 for a similar calculus for  $\alpha = 2$  ).  $\square$



**Corollary 2.** *Let  $\alpha > 1$ ,  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$ ,  $p \geq 1$  such that  $\sup_y \mathbb{E}|u^0(y)|^q < \infty$  for some  $q \geq 2$ , and let the coefficients  $f, h_k$   $k = 0, 1, \dots, m+1$  satisfy the conditions (2) such that  $a_k \in L_\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ ,  $k = 0, 1, \dots, m+1$ . Then the equation in (1) admits a unique  $L^2$ -mild solution which is continuous in space and in time and satisfies the inequality*

$$(17) \quad \max\left\{\sup_{[0,T]} \mathbb{E}|u(s)|_2^p, \sup_{[0,T]} \sup_y \mathbb{E}|u(s, y)|^q\right\} < \infty.$$

The corollary follows from Theorem 1 in [14] and Theorem 1 in this paper.

### 3. EQUIVALENCE OF SOLUTIONS

Let us consider two kinds of solutions of variational type of the SFPDE in (1), for which the coefficients  $f, h_k$  satisfy the conditions (2) and the initial condition  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$ .

**Definition 3.** *A  $L^2$ -valued  $\mathcal{F}_t$ -adapted stochastic process  $u = \{u(t, \cdot), t \in [0, T]\}$  is said to be a weak solution of the first kind on the interval  $[0, T]$  of Equation (1), if  $u$  satisfies the assumption (7) and the following integral equation, for all  $t \in [0, T]$  and for all  $\phi \in C_0^\infty$ , where  $\phi \in C_0^\infty$  is the set of infinitely differentiable functions with compact support on  $\mathbb{R}$ :*

$$(18) \quad \begin{aligned} \int_{\mathbb{R}} u(t, x) \phi(x) dx &= \int_{\mathbb{R}} u^0(x) \phi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) D_{-\delta}^\alpha \phi(x) dx ds \\ &+ \sum_{k=0}^m (-1)^k \int_0^t \int_{\mathbb{R}} h_k(s, x, u(s, x)) \phi^{(k)}(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} f(s, x, u(s, x)) \phi(x) W(dx ds) \quad a.s. \end{aligned}$$

**Definition 4.** *A  $L^2$ -valued  $\mathcal{F}_t$ -adapted stochastic process  $u = \{u(t, \cdot), t \in [0, T]\}$  is said to be a weak solution of the second kind on the interval  $[0, T]$  of Equation (1), if  $u$  satisfies the assumption (7) and the following integral equation, for all  $t \in [0, T]$  and for all  $\psi \in C^{1,\infty}((0, t) \times \mathbb{R})$  and such that  $\psi(s, \cdot) \in D(D_\delta^\alpha), \forall s < t$ :*

$$(19) \quad \begin{aligned} \int_{\mathbb{R}} u(t, x) \psi(t, x) dx &= \int_{\mathbb{R}} u^0(x) \psi(0, x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_s \psi(s, x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} u(s, x) D_{-\delta}^\alpha \psi(s, x) dx ds \\ &+ \sum_{k=0}^m (-1)^k \int_0^t \int_{\mathbb{R}} h_k(s, x, u(s, x)) \partial_x^{(k)} \psi(s, x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} f(s, x, u(s, x)) \psi(s, x) W(dx ds) \quad a.s. \end{aligned}$$

**Theorem 2.** *For  $p \geq 2$ , the different notions of solutions given in Definitions 2, 3 and 4 are equivalent.*

*Proof.* We prove the equivalence between Definition 2 and Definition 4. The other equivalences are obtained by a similar way ( see [18], for  $\alpha = 2$ ).

Let  $u = \{u(t), t \in [0, T]\}$  be a weak solution of second kind of Equation (1) and let  $\phi(\cdot) \in C_0^\infty(\mathbb{R})$ . We define the function  $\psi^t(s, x)$  by

$$\psi^t(s, x) = \begin{cases} \phi(x), & \text{when } t = s, \\ \int_{\mathbb{R}} -\delta G_\alpha(t-s, x-y) \phi(y) dy, & \text{when } s < t, \end{cases}$$

The function  $\psi^t(.,.) \in C^{1,\infty}((0,t) \times \mathbb{R})$  and we have for all fixed  $s < t$ ,  $\mathcal{F}\{\psi^t(s,x), \lambda\} = e^{-\delta\psi_\alpha(\lambda)(t-s)}\hat{\phi}(\lambda)$ . Hence  $\psi^t(s,x) \in D({}_x D_{-\delta}^\alpha)$  and  ${}_x D_{-\delta}^\alpha \psi^t(s,x) = \mathcal{F}^{-1}\{-\delta\psi_\alpha(\lambda)e^{-\delta\psi_\alpha(\lambda)(t-s)}\hat{\phi}(\lambda), x\}$ . On the other hand for fixed  $x$ ,  $\psi^t(s,x)$  is differentiable with respect to  $s < t$ , because  ${}_{-\delta}G_\alpha(t-s, z)$  is differentiable with respect to  $s$  and  $\delta_{s-\delta}G_\alpha(t-s, .)\phi(.)$  is integrable. Further

$$\begin{aligned} \partial_s \psi^t(s,x) &= \int_{\mathbb{R}} \partial_{s-\delta} G_\alpha(t-s, x-y) \phi(y) dy \\ &= - \int_{\mathbb{R}} \partial_{\tau-\delta} G_\alpha(\tau, y)|_{\tau=t-s} \phi(x-y) dy \\ &= - \int_{\mathbb{R}} {}_y D_{-\delta-\delta}^\alpha G_\alpha(t-s, y) \phi(x-y) dy \\ &= -\mathcal{F}^{-1}\{-\delta\psi_\alpha(\lambda)e^{-\delta\psi_\alpha(\lambda)(t-s)}\hat{\phi}(\lambda), x\} \\ &= -{}_x D_{-\delta}^\alpha \psi^t(s,x). \end{aligned}$$

We replace  $\psi(s,x)$  by  $\psi^t(s,x)$  in equation (19), apply deterministic and stochastic Fubini's Theorems and the fact that  ${}_{-\delta}G_\alpha(t-s, x-y) = {}_\delta G_\alpha(t-s, y-x)$ . We interpret the integrals on  $\mathbb{R}$  as the scalar product in  $L^2(\mathbb{R})$  and use estimates as in section 2 to prove that  $\mathcal{A}_k u(t) \in L^2(\mathbb{R})$  a.s.,  $\forall k \in \{0, 1, \dots, m+2\}$ , we get  $\langle u(t) - (\mathcal{A}_0 + \sum_{k=0}^m \mathcal{A}_{k+1} - \mathcal{A}_{m+2})u(t), \phi \rangle_{L^2} = 0$ . Since  $C_0^\infty$  is dense in  $L^2(\mathbb{R})$ , we obtain that  $u = \{u(t), t \in [0, T]\}$  is a mild solution of Equation (1). Fubini's Theorems are applied thanks to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u^0(x)| {}_\delta G_\alpha(t-s, y-x) |\phi(y)| dy dx \leq |u^0|_2 |\phi|_2 |{}_\delta G_\alpha(1, .)|_2 < \infty \quad a.s.,$$

$$\begin{aligned} \mathbb{E} \left( \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |h_k(s, x, u(s, x))| |{}_{-\delta} G_\alpha^{(k)}(t-s, x-y)| |\phi(y)| dy dx ds \right)^p \\ \leq \mathbb{E} \left( \int_0^t |h_k(s, ., u(s, .))|_2 |{}_{-\delta} G_\alpha^{(k)}(t-s, .)| * |\phi|_2 ds \right)^p \\ \leq K |\phi|_2^p \mathbb{E} \left( 1 + \int_0^t (t-s)^{-\frac{k}{\alpha}} |u(s)|_2 ds \right)^p \\ \leq K |\phi|_2^p \mathbb{E} \left( 1 + \int_0^t (t-s)^{-\frac{k}{\alpha}} |u(s)|_2^p ds \right) \\ \leq K (1 + \sup_{[0,T]} \mathbb{E} |u(s)|_2^p) < \infty, \end{aligned}$$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \mathbb{E} \left( \int_{\mathbb{R}} \phi(y) f(s, x, u(s, x)) {}_{-\delta} G_\alpha(t-s, x-y) dy \right)^2 dx ds \\ \leq K |\phi|_\infty^2 |{}_{-\delta} G_\alpha(1, .)|_1^2 \mathbb{E} (1 + \int_0^t |u(s)|_2^2 ds) \\ \leq K \mathbb{E} (1 + \int_0^t |u(s)|_2^2 ds). \end{aligned}$$

Using Jensen inequality, we get

$$\begin{aligned} & \left( \int_0^t \int_{\mathbb{R}} \mathbb{E} \left( \int_{\mathbb{R}} \phi(y) f(s, x, u(s, x)) \delta G_{\alpha}(t-s, x-y) dy \right)^2 dx ds \right)^{\frac{p}{2}} \\ & \leq K \left( 1 + \int_0^t \mathbb{E} |u(s)|_2^p ds \right) < \infty. \end{aligned}$$

Let now  $u = \{u(t), t \geq 0\}$  be a mild solution of Equation (1). From Theorem 1,  $u$  satisfies the inequality (7). Furthermore, for fixed  $t > 0$ , let  $\psi \in C^{1,\infty}([0, t] \times \mathbb{R})$  such that  $\psi(s, \cdot) \in D(D_{\delta}^{\alpha}), \forall s < t$ . by replacing the right hand side of (6) in the left hand side of (19), we get

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \psi(t, x) dx &= \int_{\mathbb{R}} \psi(t, x) \left( \int_{\mathbb{R}} \delta G_{\alpha}(t, x-y) u^0(y) dy \right) dx \\ &+ \sum_{k=0}^m (-1)^k \int_{\mathbb{R}} \psi(t, x) \left( \int_0^t \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(t-s, x-y) h_k(s, y, u(s, y)) dy ds \right) dx \\ &+ \int_{\mathbb{R}} \psi(t, x) \left( \int_0^t \int_{\mathbb{R}} \delta G_{\alpha}(t-s, x-y) f(s, y, u(s, y)) W(dy ds) \right) dx. \end{aligned} \quad (20)$$

Applying Fubini's Theorems (deterministic and stochastic) to each term on the right hand of (20), we get the terms  $\int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(t-s, x-y) \psi(t, x) dx, k = 0, 1, \dots, m$  in the right hand side of (20). Using the properties of Green's function  $\delta G_{\alpha}(t, x)$  and the integral by parts (see [13]), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \delta G_{\alpha}(t-s, x-y) \psi(t, x) dx &= \psi(s, y) + \int_{\mathbb{R}} \int_s^t \frac{\partial}{\partial \sigma} (\delta G_{\alpha}(\sigma-s, x-y) \psi(\sigma, x)) dx d\sigma \\ &= \psi(s, y) + \int_s^t \int_{\mathbb{R}} \delta G_{\alpha}(\sigma-s, x-y) {}_x D_{-\delta}^{\alpha} \psi(\sigma, x) dx d\sigma \\ &+ \int_s^t \int_{\mathbb{R}} \delta G_{\alpha}(\sigma-s, x-y) \partial_{\sigma} \psi(\sigma, x) dx d\sigma. \end{aligned} \quad (21)$$

Furthermore, by the commutativity of the operators  ${}_x D_{-\delta}^{\alpha}$  and  $D^k$ , where this last is the classical differential operator of entire order  $k$ , we get

$$\begin{aligned} \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(t-s, x-y) \psi(t, x) dx &= (-1)^k \int_{\mathbb{R}} \delta G_{\alpha}(t-s, x-y) \psi_x^{(k)}(t, x) dx \\ &= \psi_y^{(k)}(s, y) + \int_s^t \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(\sigma-s, x-y) {}_x D_{-\delta}^{\alpha} \psi(\sigma, x) dx d\sigma \\ &+ \int_s^t \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(\sigma-s, x-y) \partial_{\sigma} \psi(\sigma, x) dx d\sigma. \end{aligned} \quad (22)$$

By replacing (21) and (22) in to corresponding terms in (20) and again by applying Fubini's Theorem, we get

$$\begin{aligned} \int_{\mathbb{R}} \psi(t, x) \left( \int_{\mathbb{R}} \delta G_{\alpha}(t, x - y) u^0(y) dy \right) dx &= \int_{\mathbb{R}} u^0(y) \psi(0, y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \delta G_{\alpha}(\sigma, x - y) u^0(y) dy \right) ({}_x D_{-\delta}^{\alpha} \psi(\sigma, x) + \partial_{\sigma} \psi(\sigma, x)) dx d\sigma, \end{aligned} \quad (23)$$

$$\begin{aligned} \int_{\mathbb{R}} \psi(t, x) \left( \int_0^t \int_{\mathbb{R}} \delta G_{\alpha}(t - s, x - y) f(s, y, u(s, y)) W(dy ds) \right) dx &= \int_0^t \int_{\mathbb{R}} f(s, y, u(s, y)) \psi(s, y) W(dy ds) \\ &+ \int_0^t \int_{\mathbb{R}} \left( \int_0^{\sigma} \int_{\mathbb{R}} \delta G_{\alpha}(\sigma - s, x - y) f(s, y, u(s, y)) W(dy ds) \right) ({}_x D_{-\delta}^{\alpha} \psi(\sigma, x) + \partial_{\sigma} \psi(\sigma, x)) dx d\sigma, \end{aligned} \quad (24)$$

$$\begin{aligned} \int_{\mathbb{R}} \psi(t, x) \left( \int_0^t \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(t - s, x - y) h_k(s, y, u(s, y)) dy ds \right) dx &= \int_0^t \int_{\mathbb{R}} h_k(s, y, u(s, y)) \psi_y^{(k)}(s, y) dy ds \\ &+ \int_0^t \int_{\mathbb{R}} \left( \int_0^{\sigma} \int_{\mathbb{R}} \delta G_{\alpha}^{(k)}(\sigma - s, x - y) h_k(s, y, u(s, y)) dy ds \right) ({}_x D_{-\delta}^{\alpha} \psi(\sigma, x) + \partial_{\sigma} \psi(\sigma, x)) dx d\sigma. \end{aligned} \quad (25)$$

Replacing these equalities in (20) and using the fact that  $u(t, \cdot)$  is a mild solution, we get that  $u(t, \cdot)$  satisfies Equation (19). We can check as in the first part of the prove that Fubini's Theorems can be applied.  $\square$

As a consequence of Theorems 1 and 2, we have

**Corollary 3.** *Equation (1) for with the coefficients  $f, h_k$  satisfying the conditions (2) and the initial condition  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$ ,  $p \geq 2$  admits a unique weak solution of first kind and of second kind.*

#### 4. APPLICATION OF FOURIER TRANSFORM IN A RANDOM SFPDE

We consider the following SFPDE obtained from Equation (1) by taking  $h_k(t, x, r) = c_k r$ , for all  $0 \leq k \leq m$  and where  $f$  may be random but independent of the solution  $u$

$$(26) \quad \frac{\partial}{\partial s} u(s, y) = {}_x D_{-\delta}^{\alpha} u(s, y) + \sum_0^m c_k \frac{\partial^k}{\partial y^k} u(s, y) + f(s, y) \frac{\partial^2 W}{\partial s \partial y}(y, s).$$

Multiplying the two sides of the above equation by  $e^{i\lambda y}$  and integrating with respect to  $s$  and to  $y$ , we get

$$(27) \quad \hat{u}(t, \lambda) = \hat{u}^0(\lambda) + ({}_s \psi_{\alpha}(\lambda) + \sum_0^m c_k (-i\lambda)^k) \int_0^t \hat{u}(s, \lambda) ds + \int_0^t \int_{\mathbb{R}} e^{i\lambda y} f(s, y) W(dy ds)$$

which has a rigorous meaning. Let us denote by  $\eta_{\lambda}(t)$  the stochastic integral in the equality above. It is known that  $\eta_{\lambda}(t)$  is a martingale [30]. The Equation (27) is

equivalent to the following linear stochastic differential equation perturbed by the martingale  $\{\eta_\lambda(t), t \geq 0\}$

$$(28) \quad d\hat{u}_\lambda(t) = (\delta\psi_\alpha(\lambda) + \sum_0^m c_k(-i\lambda)^k)\hat{u}_\lambda(t)dt + d\eta_\lambda(t),$$

with initial condition  $\hat{u}_\lambda(t) = \hat{u}^0(\lambda)$ . This equation admits the explicit solution

$$(29) \quad \hat{u}_\lambda(t) = \hat{u}_0(\lambda)e^{(\delta\psi_\alpha(\lambda) + \sum_0^m c_k(-i\lambda)^k)t} + \int_0^t e^{(\delta\psi_\alpha(\lambda) + \sum_0^m c_k(-i\lambda)^k)(t-s)}d\eta_\lambda(s),$$

which is a generalization of an Ornstein-Uhlenbeck process.

**Proposition 1.** *Consider the equation (26), with  $c_k = 0, \forall k \in \overline{0m}$  and  $u^0 \in L^p(\Omega, \mathcal{F}_0, L^2)$ . The process whose Fourier transform is given by (29) is equivalent to the solution in the sense of Definitions 2, 3 and 4.*

*Proof.* Let  $\{u(t), t \geq 0\}$  be the process whose Fourier transform is given by (29). It is sufficient to prove that it is the mild solution. Then  $\hat{u}(t, \lambda)$  is given for all  $\lambda$  by the following formulae

$$(30) \quad \hat{u}(t, \lambda) = \hat{u}^0(\lambda)e^{\delta\psi_\alpha(\lambda)t} + \int_0^t e^{\delta\psi_\alpha(\lambda)(t-s)}d\eta_\lambda(s).$$

Applying the inverse Fourier transform on the two sides of this equation, replacing  $\eta_\lambda(s)$  by its values and using the Fubini's Theorem, we get

$$\begin{aligned} u(t, \cdot) &= u^0 * \delta G_\alpha(t, \cdot) + \int_0^t \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-i\lambda(\cdot-y)} e^{\delta\psi_\alpha(\lambda)(t-s)} d\lambda \right) f(s, y) W(dy ds) \\ &= \int_{\mathbb{R}} \delta G_\alpha(t, \cdot - y) u^0(y) dy + \int_0^t \int_{\mathbb{R}} \delta G_\alpha(t - s, \cdot - y) f(s, y) W(dy ds). \end{aligned}$$

By the same calculus and using the Fourier transform, we prove that the Fourier transform of the mild solution is given by (30).  $\square$

#### REFERENCES

- [1] Angulo, J.M and Ruiz-Medina, M.D and Anh, V.V and Grecksch, W. *Fractional Diffusion and Fractional Heat Equation*. Adv.Appl.Prob. 32, 1077–1099 (2000).
- [2] Anh, V. V. and Heyde, V.V and Leonenko, N.N. *Dynamic Models of Long-Memory Processes Driven by Lévy Noise*. J. Appl. Prob. 39, 730–747 (2002).
- [3] Benson, A.D. and Wheatcraft, W. S. and Meerschaert, M. M. *The Fractional-Order Governing Equation of Lévy Motion*. Water Resources Research 36, no 6, 1413–1423 (200).
- [4] Biler, P. and Funaki, T. and Woyczynski, W. A. *Fractal Burgers' equations*. J. Differential Equations 148, 9–46 (1998).
- [5] Brzeźniak, Z. and L. Debbi *On Stochastic Burgers Equation Driven by a Fractional Power of the Laplacian and space-time white noise*. Stochastic Differential Equation: Theory and Applications, A volume in Honor of Professor Boris L. Rozovskii. Edited by P. H. Baxendale and S. V. Lototsky 135–167 (2007).
- [6] Caffarelli, L.A. *Some nonlinear problems involving non-local diffusions*. ICIAM 07-6th Intern. Congress on Industrial and Applied Math. Eur. Math. Soc. Zurich 43–56 (2009).
- [7] Caffarelli, L.A. and Vasseur, A. *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*. Ann. of Math. 2, 171 no. 3, 1903–1930 (2010).
- [8] Chojnowska-Michalik, A. *Stochastic Differential Equations In Hilbert Spaces*. Probability Theory, Banach Publications 5, 53–74 (1979).

- [9] Da Prato, G. and Zabczyk, J. *Stochastic Equations in Infinite Dimensions*. Springer, Cambridge University Press 1992.
- [10] Dalang, R. and Mueller, C. *Some Non-linear S.P.D.E.'s That are Second Order in Time*. Electronic Journal of Probability. 8 no. 1, 1–21 (2003).
- [11] Dalang, R. and Sanz-Solé, M. *Regularity of the sample paths of a class of second-order spde's*. J. Funct. Anal. 227, no. 2 304–337 (2005).
- [12] Debbi, L. *Explicit Solutions of Some Fractional Equations Via stable subordinators*. J. Appl. Math. Stoch. Anal. ID 93502, 18pp (2006).
- [13] Debbi, L. *On Some Properties of a High Order Fractional Differential Operator Which is Not in General Selfadjoint*. Appl. Math. Sci. 1 no. 25-28, 1325-1339 (2007).
- [14] Debbi, L. and Dozzi, M. *On The Solution of Non Linear Stochastic Fractional Partial Differential Equations in one Spatial Dimension*. Stochastic Processes and Their Applications 115 no. 11, 1764–1781 (2005).
- [15] Funaki, T. *Probabilistic Construction of the Solution of Some Higher Order Parabolic Differential Equation*. Proc. Japan. Acad. Ser.A 55, 176–179 (1979).
- [16] Giona, M. and Roman, E. *Fractional Diffusion Equation On Fractals: One-dimensional Case and Asymptotic Behaviour*. J. Phys.A: Math. Gen. 25, 2093–2105 (1992).
- [17] Gorenflo, R. and Mainardi, F. *Random Walk Models for Space-Fractional Diffusion Processes*. Fractional Calculus & Applied Analysis, 1 no 2, 167–191 (1998).
- [18] Gyöngy, I. *On The Stochastic Burgers Equations*. Preprint 15 (1996).
- [19] Gyöngy, I. and Nualart, D. *On the Stochastic Burgers' Equation in the Real Line*. The Annals of Probability, 27 no 2, 782–802 (1999).
- [20] Gyöngy, I. *Existence and Uniqueness Results for Semilinear Stochastic Partial Differential Equations*. Stochastic Processes and Their Applications, 73, 271–299 (1998).
- [21] Hu, Y. *Heat Equations with Fractional White noise Potentials*. Appl. Math. Optim 43, 221–243 (2001).
- [22] Jacob, N. *Pseudo Differential Operators, Markov processes I*. Imperial College Press 2001.
- [23] Jacob, N. *Pseudo Differential Operators, Markov processes II*. Imperial College Press (2002).
- [24] Léon, A. J. *Stochastic Evolution Equations With Respect to Semimartingales In Hilbert Spaces*. Stochastics and Stochastics Reports 27, 1–21 (1989).
- [25] Mueller, C. *The Heat Equation with Lévy noise*. Stoch. Proc. Appl. 74, 67–82 (1998).
- [26] Nualart, D. and Rascanu, A. *Differential Equations Driven by Fractional Brownian Motion*. Collectanea Mathematica, 53, 55–81 (2002).
- [27] Nualart, D. and Vuillermot, P. A. *Variational solutions for a class of fractional stochastic partial differential equations*. C. R. Math. Acad. Sci. Paris, 340, 281–286 (2005).
- [28] Sanz-Solé, M. and Vuillermot, P. A. *Equivalence and Hölder-Sobolev regularity of stochastic partial differential equations*. Ann. I. H. Poincaré-PR39, 4, 703–742 (2003).
- [29] Uchaikin, V.V. and Zolotarev, V. M. *Chance and Stability, Stable Distributions and their Applications*. Modern Probability and Statistics, VSP. (1999).
- [30] Walsh, J. B. *An introduction to stochastic partial differential equations*. Lectures Notes in Mathematics 1180 Ecole d'été de Probabilités de Saint-Flour XIV-1984, Springer-Verlag 265–439 (1986).
- [31] Wu, J.L. *Fractal Burgers equation with stable Lévy noise*. International Conference SPDE and Applications-VII. January 4-10, Levico Term Trento, Italy 2004.
- [32] Zabczyk, J. *The Fractional Calculus and Stochastic Evolution Equations*. Progress in Probability, Barcelona Seminar on Stochastic Analysis 32, 222–234 (1993).
- [33] Zaslavsky, M. G. *Renormalization Group Theory of Anomalous Transport in Systems with Hamiltonian Chaos*. Chaos 4, no. 1, 25–33 (1994).
- [34] Zaslavsky, M. G. *Multifractal Kinetics*. Physica A 288, 431–443 (2000).
- [35] Zaslavsky, M. G. and Abdullaev, S. S. *Scaling Property and Anomalous Transport of Particles Inside the Stochastic Layer*. Physica Review E 51, no. 5, 3901–3910 (1995).

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